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# A derivation of the BRST operator for non-critical W-strings

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We derive the recently proposed BRST charge for non-critical W strings from a lagrangian approach. The basic observation is that, despite appearances, the combination of two classical “matter” and “Toda”  $w_3$  systems leads to a closed modified gauge algebra, which is of the so-called soft type. Based on these observations, a novel way to construct critical  $W_3$  strings is given.

## 1. Introduction

So far, all attempts to construct *critical*  $W_3$  strings [1,2] have concentrated on the standard  $W_3$  algebra of Zamolodchikov [3]. The corresponding BRST charge, which plays an essential role in the determination of the physical state spectrum, was constructed some time ago by Thiery-Mieg [4]. This result has been generalized in ref. [5] to arbitrary quadratically generated non-linear algebras, while in ref. [6] the BRST charge was derived from a lagrangian point of view. The construction of the BRST charge involves the introduction of ghosts and antighosts for the Virasoro and the spin-3 symmetries. It turns out that nilpotency of the BRST charge requires that the matter contribution  $c_M$  to the central charge be  $c_M=100$ . One therefore has to search for matter realizations of the  $W_3$  symmetry at this value of the central charge. The standard realization of Fateev and Zamolodchikov [7] involves only two free scalars, but more would be preferred in view of string applications. In ref. [8], realizations involving an arbitrary number of scalars were given. The application of this multi-scalar realization in the context of critical  $W_3$  strings was discussed in ref. [2].

Recently, Bershadsky, Lerche, Nemeschansky and Warner [9] constructed a BRST charge for a *non-critical*  $W_3$  string, i.e. for a  $W_3$  minimal model coupled to  $W_3$  gravity. This BRST charge differs from the one of ref. [4]. The construction of ref. [9] is an important step forwards in the study of non-critical  $W_3$  strings, which are expected to be exactly solvable. An earlier investigation of non critical  $W_3$  strings was given in refs. [10,11], where both the induced and the effective  $W_3$ -gravity lagrangians were obtained and the  $W_3$  KPZ formula was derived. In the analysis of ref. [9] one starts with two mutually commuting  $W_3$  algebras generated by the energy-momentum tensors  $T^{(i)}$  and the spin-3 currents  $W^{(i)}$  ( $i=1, 2$ ). The  $i=1$  copy represents the matter sector while the  $i=2$  copy represents the  $W_3$ -Liouville (Toda) sector, which does not decouple in the non-critical case. Both satisfy the quantum  $W_3$  algebra with operator product expansions (OPE) given by

$$\begin{aligned} T^{(i)}(z)T^{(i)}(w) &\sim \frac{\frac{1}{2}c_i}{(z-w)^4} + \frac{2T^{(i)}}{(z-w)^2} + \frac{\partial T^{(i)}}{z-w}, \\ T^{(i)}(z)W^{(i)}(w) &\sim \frac{3W^{(i)}}{(z-w)^2} + \frac{\partial W^{(i)}}{z-w}, \end{aligned} \quad (1)$$

$$W^{(i)}(z)W^{(i)}(w) \sim \frac{c_i}{9\beta_i^2(z-w)^6} + \frac{2T^{(i)}}{3\beta_i^2(z-w)^4} + \frac{\partial T^{(i)}}{3\beta_i^2(z-w)^3} \\ + \frac{1}{(z-w)^2} \left( \frac{2}{3}A^{(i)} + \frac{1}{10\beta_i^2} \partial^2 T^{(i)} \right) + \frac{1}{z-w} \left( \frac{1}{3} \partial A^{(i)} + \frac{1}{45\beta_i^2} \partial^3 T^{(i)} \right), \quad (1 \text{ cont'd})$$

where

$$\beta_i^2 \equiv \frac{16}{5c_i + 22}, \quad (2)$$

and

$$A^{(i)} = (T^{(i)}T^{(i)}) - \frac{3}{10}\partial^2 T^{(i)}. \quad (3)$$

For later purposes, it is convenient to use a non-standard normalization of the spin-3 generators such that the coefficient in front of the composite  $A$ -term above is independent of the central charge.

The sum of the matter and Liouville system, i.e.  $T^{\text{tot}} = T^{(1)} + T^{(2)}$  and  $W^{\text{tot}} = W^{(1)} + W^{(2)}$ , does not form a closed  $W_3$  algebra. One may verify that there is no way to construct a  $W_3$  algebra out of  $T^{(i)}$  and  $W^{(i)}$  only. The remarkable result of ref. [9] is that nevertheless it is possible to construct a nilpotent BRST charge for the matter and Liouville systems. Introduce ghosts and antighosts  $c^{(1)}, b^{(1)}, c^{(2)}, b^{(2)}$  for the Virasoro and the spin-3 symmetry, respectively. The BRST charge of ref. [9] is then given by

$$Q = \frac{1}{2\pi i} \oint [c^{(1)}(T^{(1)} + T^{(2)} + \frac{1}{2}T_{\text{gh}}) + c^{(2)}(W^{(1)} \pm iW^{(2)} + \frac{1}{2}W_{\text{gh}})], \quad (4)$$

where  $T_{\text{gh}}$  is the energy-momentum tensor and  $W_{\text{gh}}$  the spin-3 current of the ghost-antighost system<sup>#1</sup>. The BRST charge is nilpotent, provided that  $c_1 + c_2 = 100$ .

To obtain a better understanding of the basic difference between the BRST charges of refs [4,9], it is instructive to restrict oneself to the classical  $w_3$  algebra. The classical  $w_3$  algebra is obtained from the quantum  $W_3$  algebra by omitting in (1) all central terms and by retaining, in the OPE of the two spin-3 generators, the quadratic terms only. In the approach of ref. [4] it is assumed that the classical  $w_3$  algebra is represented by the matter sector only, and the BRST charge corresponding to such a  $w_3$  algebra is constructed. On the other hand, in the approach of ref. [9], there is a matter and Liouville system, corresponding to the systems (1) and (2), respectively, which separately satisfy the  $w_3$  algebra. We assume that  $T^{(i)}$  and  $W^{(1)}$  are real and that  $W^{(2)}$  is imaginary<sup>#2</sup>. The (real) sum of the Liouville and matter systems, i.e.  $T^{(1)} + T^{(2)}$  and  $W^{(1)} + iW^{(2)}$ , does not satisfy the  $w_3$  algebra, as in the quantum case. The important point however is that, unlike the quantum case, the sum here does satisfy a *modified*  $w_3$  algebra. The difference between the two algebras resides in the quadratic terms. The matter system satisfies a  $w_3$  algebra where the quadratic terms are proportional to  $+T^{(1)}T^{(1)}$  while the Liouville sector, using the *real* generators  $T^{(2)}$  and  $iW^{(2)}$ , satisfies a  $w_3$  algebra with an opposite sign of the quadratic terms, i.e. they are proportional to  $-T^{(2)}T^{(2)}$ . Therefore, the sum satisfies a modified  $w_3$ -algebra where the quadratic terms are proportional to  $(T^{(1)})^2 - (T^{(2)})^2$ , i.e.

$$(W^{(1)} + iW^{(2)})(z)(W^{(1)} + iW^{(2)})(w) \sim (T^{(1)} - T^{(2)})(T^{(1)} + T^{(2)}). \quad (5)$$

Since the quadratic terms are proportional to the total spin-2 generator  $T^{(1)} + T^{(2)}$ , one is left with a *closed* gauge algebra which is of the so-called soft type. In particular, we see that the combination  $T^{(1)} - T^{(2)}$  occurs as a field-dependent structure constant of the modified algebra. The algebra is also open in the sense that it closes

<sup>#1</sup> Later we will derive the explicit expressions for  $T_{\text{gh}}$  and  $W_{\text{gh}}$ ;  $W_{\text{gh}}$  will also depend on  $T^{(1)}$  and  $T^{(2)}$ .

<sup>#2</sup> Following the literature, we use in this paper the convention that the matter fields are imaginary and the Liouville fields are real. The Liouville system only satisfies the  $w_3$  algebra using an *imaginary* spin-3 generator. Rescaling with a factor  $i$  leads to a real spin-3 generator but also changes the sign of the quadratic terms in the algebra.

only on-shell, with the field equations given by the spin-2 and spin-3 constraints  $T^{(1)} + T^{(2)} = 0$  and  $W^{(1)} + iW^{(2)} = 0$ , respectively.

It is the aim of this letter to show how the BRST charge of ref. [9] follows from the abovementioned modified  $w_3$  algebra. We will first perform a classical gauging of this modified algebra and then quantize the system. We thus derive the results of ref. [9] from a lagrangian point of view.

## 2. Gauging

As our starting point we consider two sets of two scalar fields, an imaginary one  $\varphi^{(1)} = \sum_{r=1}^2 \varphi_r^{(1)} H_r$  and a real one  $\varphi^{(2)} = \sum_{r=1}^2 \varphi_r^{(2)} H_r$ ,<sup>#3</sup> where

$$H_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad H_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}. \quad (6)$$

The free-field action

$$S_0 = \sum_{i=1}^2 \frac{1}{2} \int \text{tr}(\partial\varphi^{(i)} \bar{\partial}\varphi^{(i)}) \quad (7)$$

transforms under the classical  $w_3$  transformations given by

$$\begin{aligned} \delta\varphi^{(1)} &= \epsilon \partial\varphi^{(1)} + i\lambda \partial\varphi^{(1)}\varphi^{(1)} - \frac{1}{3}i\lambda \text{tr}(\partial\varphi^{(1)} \partial\varphi^{(1)}), \\ \delta\varphi^{(2)} &= \epsilon \partial\varphi^{(2)} \mp \lambda \partial\varphi^{(2)}\varphi^{(2)} \pm \frac{1}{3}\lambda \text{tr}(\partial\varphi^{(2)} \partial\varphi^{(2)}), \end{aligned} \quad (8)$$

as

$$\delta S_0 = - \int \bar{\partial}\epsilon(T^{(1)} + T^{(2)}) - \int \bar{\partial}\lambda(W^{(1)} \pm iW^{(2)}), \quad (9)$$

where

$$T^{(i)} = -\frac{1}{2} \text{tr}(\partial\varphi^{(i)} \partial\varphi^{(i)}), \quad W^{(i)} = -\frac{1}{3}i \text{tr}(\partial\varphi^{(i)} \partial\varphi^{(i)} \partial\varphi^{(i)}) \quad (10)$$

are the spin-2 and spin-3 Noether currents. The Poisson bracket algebra of  $\{T^{(1)}, W^{(1)}\}$  and  $\{T^{(2)}, W^{(2)}\}$  separately is given by the  $w_3$  algebra. The Noether currents  $T^{(i)}$  and  $W^{(i)}$  transform as

$$\begin{aligned} \delta T^{(1)} &= \epsilon \partial T^{(1)} + 2 \partial\epsilon T^{(1)} + 2\lambda \partial W^{(1)} + 3 \partial\lambda W^{(1)}, \\ \delta T^{(2)} &= \epsilon \partial T^{(2)} + 2 \partial\epsilon T^{(2)} \pm 2i\lambda \partial W^{(1)} \pm 3i \partial\lambda W^{(1)}, \\ \delta W^{(1)} &= \epsilon \partial W^{(1)} + 3 \partial\epsilon W^{(1)} + \frac{1}{3}\lambda \partial(T^{(1)}T^{(1)}) + \frac{2}{3}\partial\lambda(T^{(1)}T^{(1)}), \\ \delta W^{(2)} &= \epsilon \partial W^{(2)} + 3 \partial\epsilon W^{(2)} \pm \frac{1}{3}i\lambda \partial(T^{(2)}T^{(2)}) \pm \frac{2}{3}i \partial\lambda(T^{(2)}T^{(2)}), \end{aligned} \quad (11)$$

where we have used the fact that

$$\text{tr}(\partial\varphi)^4 = \frac{1}{2} [\text{tr}(\partial\varphi)^2]^2. \quad (12)$$

The factors of  $i$  in the first line of (8) reflect the fact that  $\varphi^{(1)}$  is imaginary. The  $\pm$  signs in the second line of (8) are related to the invariance of the  $w_3$  algebra under the replacement  $W \rightarrow -W$  of the spin-3 generator. This will later on explain the two possible combinations in the BRST charge of ref. [9].

<sup>#3</sup> One can associate  $\varphi^{(1)}$  with the (imaginary) matter fields and  $\varphi^{(2)}$  with the (real) Liouville fields. In a non-critical  $W_3$ -string the Liouville fields describe a  $SL(3)$  Toda system.

The action becomes invariant under (8), if a minimal coupling term  $S_1$  is added to the free action  $S_0$ :

$$S_1 = \int h_{(2)}(T^{(1)} + T^{(2)}) + \int h_{(3)}(W^{(1)} \pm iW^{(2)}), \quad (13)$$

where  $h_{(2)}$  is the Beltrami differential and  $h_{(3)}$  is its spin-3 generalization<sup>#4</sup>. Indeed, if  $h_{(2)}$  and  $h_{(3)}$  transform as

$$\begin{aligned} \delta h_{(2)} &= \bar{\delta}\epsilon + \epsilon \partial h_{(2)} - \partial\epsilon h_{(2)} + \frac{1}{3}(\lambda \partial h_{(3)} - \partial\lambda h_{(3)})(T^{(1)} - T^{(2)}), \\ \delta h_{(3)} &= \bar{\delta}\lambda + \epsilon \partial h_{(3)} - 2\partial\epsilon h_{(3)} + 2\lambda \partial h_{(2)} - \partial\lambda h_{(2)}, \end{aligned} \quad (14)$$

then  $S_0 + S_1$  is invariant under classical  $w_3$  gauge transformations. The fact that minimal coupling is sufficient for gauging a chiral  $w_3$  symmetry is due to Hull [6]. In ref. [12] the chiral gauge theory was generalized to the case where both the chiral and antichiral  $w_3$  symmetries were gauged. It was shown that through the introduction of auxiliary fields, the so-called nested covariant derivatives, the non-chiral gauged  $w_3$  symmetry reduces to two copies of the chiral case. It is not hard to generalize the analysis of ref. [12] to the case here at hand.

We note that the Poisson-bracket algebra of the total currents  $T^{(1)} + T^{(2)}$  and  $W^{(1)} \pm iW^{(2)}$ , occurring in (13), is given by the modified  $w_3$  algebra mentioned in the Introduction. The transformation rules of the gauge fields  $h_{(2)}$  and  $h_{(3)}$  are determined by the structure functions of this modified algebra. The structure of this algebra can be made explicit by calculating the commutators of the Virasoro and spin-3 symmetries. We find that the commutator of two Virasoro symmetries and that of a Virasoro symmetry with a spin-3 symmetry still assume the standard form

$$[\delta(\epsilon_1), \delta(\epsilon_2)] = \delta(\epsilon_3 = \epsilon_2 \partial\epsilon_1 - \epsilon_1 \partial\epsilon_2), \quad [\delta(\epsilon_1), \delta(\lambda_2)] = \delta(\lambda_3 = 2\lambda_2 \partial\epsilon_1 - \epsilon_1 \partial\lambda_2). \quad (15)$$

The difference with the usual  $w_3$  algebra is only manifest in the commutator of two spin-3 transformations. We find that this commutator is given by

$$\begin{aligned} [\delta(\lambda_1), \delta(\lambda_2)]\varphi^{(1)} &= \delta(\epsilon_3)\varphi^{(1)} + \frac{1}{3}(\lambda_2 \partial\lambda_1 - \lambda_1 \partial\lambda_2) \frac{\delta(S_0 + S_1)}{\delta h_{(2)}} \partial\varphi^{(1)}, \\ [\delta(\lambda_1), \delta(\lambda_2)]\varphi^{(2)} &= \delta(\epsilon_3)\varphi^{(2)} - \frac{1}{3}(\lambda_2 \partial\lambda_1 - \lambda_1 \partial\lambda_2) \frac{\delta(S_0 + S_1)}{\delta h_{(2)}} \partial\varphi^{(2)}, \\ [\delta(\lambda_1), \delta(\lambda_2)]h_{(2)} &= \delta(\epsilon_3)h_{(2)} - \frac{1}{3}(\lambda_2 \partial\lambda_1 - \lambda_1 \partial\lambda_2) \left( \partial\varphi_i^{(1)} \frac{\delta(S_0 + S_1)}{\delta\varphi_i^{(1)}} - \partial\varphi_i^{(2)} \frac{\delta(S_0 + S_1)}{\delta\varphi_i^{(2)}} \right), \\ [\delta(\lambda_1), \delta(\lambda_2)]h_{(3)} &= \delta(\epsilon_3)h_{(3)}, \end{aligned} \quad (16)$$

where

$$\epsilon_3 = \frac{1}{3}(\lambda_2 \partial\lambda_1 - \lambda_1 \partial\lambda_2)(T^{(1)} - T^{(2)}). \quad (17)$$

The combination  $T^{(1)} - T^{(2)}$ , occurring in (17), reflects the fact that it is a field-dependent structure function in the modified  $w_3$  algebra (cp. to (5)).

<sup>#4</sup> Since the Liouville fields already describe the gravity sector, one might wonder what the meaning is of the additional gauge fields  $h_{(2)}$  and  $h_{(3)}$ . The point is that a covariant field formulation of  $W_3$ -gravity is described by the spin-2 metric  $g_{\mu\nu}$  and a symmetric spin-3 tensor field  $A_{\mu\nu\rho}$ . The Liouville fields represent the conformal modes of these gauge fields while  $h_{(2)}$  and  $h_{(3)}$  represent other components of the same gravitational gauge fields.

### 3. Quantization

To describe the quantization, it is convenient to use the Batalin–Vilkovisky (BV) formalism [13]. A readable account of the BV approach can be found, for example in ref. [14]. The first step in the BV formalism consists in the introduction of extra fields, some of which are anticommuting ghost fields. At this stage, we treat the theory still at the classical level in the sense that all operations can be formulated in terms of Poisson brackets. Besides the matter fields  $\varphi^{(1)}$ , the Liouville fields  $\varphi^{(2)}$  and the gauge fields  $h_{(2)}$ ,  $h_{(3)}$ , we introduce ghost fields  $c^{(1)}$  (for the local Virasoro symmetry) and  $c^{(2)}$  (for the local spin-3 symmetry), and the corresponding anti-ghosts  $b^{(1)}$  and  $b^{(2)}$ , as well as the Nakanishi–Lautrup fields  $\pi^{(1)}$  and  $\pi^{(2)}$ . The BV formalism associates with each of these fields an antifield  $\varphi^{(1)*}$ ,  $\varphi^{(2)*}$ ,  $h_{(2)}^*$ ,  $h_{(3)}^*$ ,  $c^{(1)*}$ ,  $c^{(2)*}$ ,  $b^{(1)*}$ ,  $b^{(2)*}$ ,  $\pi^{(1)*}$  and  $\pi^{(2)*}$  of opposite statistics. The ghost number  $\text{gh}(\Phi^*)$  of the antifields  $\Phi^{* \#5}$  is given by  $\text{gh}(\Phi^*) = -1 - \text{gh}(\Phi)$ ;  $c^{(1)}$  and  $c^{(2)}$  have ghost number +1 and  $b^{(1)}$  and  $b^{(2)}$  ghost number -1, while  $\text{gh}(\pi^{(1)}) = \text{gh}(\pi^{(2)}) = 0$ . We first extend the classical action  $S_0 + S_1$  to an extended action  $S[\Phi, \Phi^*]$ , which is determined by (a)

$$S_0 + S_1 = S[\Phi, 0], \quad (18)$$

(b)  $S[\Phi, \Phi^*]$  satisfies the master equation

$$(S, S) = 2 \frac{\delta S}{\delta \Phi^A} \frac{\delta S}{\delta \Phi_A^*} = 0, \quad (19)$$

(c)  $S$  is proper, i.e. the  $2N \times 2N$  matrix  $\tilde{\partial}_\alpha \tilde{\partial}_\beta S$  has rank  $N$  on-shell.

A straightforward computation yields that

$$\begin{aligned} S[\Phi, \Phi^*] = & S_0 + S_1 + \int \text{tr} \varphi^{(1)*} (c^{(1)} \partial \varphi^{(1)} + i c^{(2)} \partial \varphi^{(1)} \partial \varphi^{(1)}) + \int \text{tr} \varphi^{(2)*} (c^{(1)} \partial \varphi^{(2)} \mp c^{(2)} \partial \varphi^{(2)} \partial \varphi^{(2)}) \\ & + \int h_{(2)}^* [\tilde{\partial} c^{(1)} + c^{(1)} \partial h_{(2)} - \partial c^{(1)} h_{(2)} + \frac{1}{3} (c^{(2)} \partial h_{(3)} - \partial c^{(2)} h_{(3)}) (T^{(1)} - T^{(2)})] \\ & + \int h_{(3)}^* (\tilde{\partial} c^{(2)} + c^{(1)} \partial h_{(3)} - 2 \partial c^{(1)} h_{(3)} + 2 c^{(2)} \partial h_{(2)} - \partial c^{(2)} h_{(2)}) \\ & - \int c^{(1)*} [c^{(1)} \partial c^{(1)} - \frac{1}{3} \partial c^{(2)} c^{(2)} (T^{(1)} - T^{(2)})] - \int c^{(2)*} (2 c^{(2)} \partial c^{(1)} + c^{(1)} \partial c^{(2)}) \\ & + \int (b^{(1)*} \pi_{(1)} + b^{(2)*} \pi_{(2)}) + \frac{1}{3} \int h_{(2)}^* \text{tr} (\varphi^{(1)*} \partial \varphi^{(1)} - \varphi^{(2)*} \partial \varphi^{(2)}) \partial c^{(2)} c^{(2)}. \end{aligned} \quad (20)$$

In order to write down the gauge-fixed action, we choose the gauge fermion

$$\Psi = \int b^{(1)} (h_{(2)} - \hat{h}_{(2)}) + \int b^{(2)} (h_{(3)} - \hat{h}_{(3)}), \quad (21)$$

where  $\hat{h}_{(2)}$  and  $\hat{h}_{(3)}$  are two *fixed*-background Beltrami differentials. The gauge-fixed action  $S_{\text{gf}}$  is now given by  $S_{\text{gf}} = S[\Phi, \Phi^*]|_\Sigma$ , where  $\Sigma$  is the hypersurface determined by

$$\Phi_A^* - \frac{\partial \Psi}{\partial \Phi^A} = 0. \quad (22)$$

The gauge-fixed action reads

<sup>#5</sup> We denote by  $\Phi^A$  all  $N$  (in our example  $N=12$ ) fields and by  $\Phi_A^*$  all antifields.

$$S_{\text{gf}} = S_0 + \int (b^{(1)} \delta c^{(1)} + b^{(2)} \delta c^{(2)}) + \int h_{(2)} (T^{(1)} + T^{(2)} + T_{\text{gh}}) \\ + \int h_{(3)} (W^{(1)} \pm iW^{(2)} + W_{\text{gh}}) + \int \pi_{(1)} (h_{(2)} - \hat{h}_{(2)}) + \int \pi_{(2)} (h_{(3)} - \hat{h}_{(3)}), \quad (23)$$

where

$$T_{\text{gh}} = -2b^{(1)} \partial c^{(1)} - \partial b^{(1)} c^{(1)} - 3b^{(2)} \partial c^{(2)} - 2\partial b^{(2)} c^{(2)}, \\ W_{\text{gh}} = -3b^{(2)} \partial c^{(1)} - \partial b^{(2)} c^{(1)} - \frac{2}{3}b^{(1)} \partial c^{(2)} (T^{(1)} - T^{(2)}) \\ - \frac{1}{3}\partial b^{(1)} c^{(2)} (T^{(1)} - T^{(2)}) - \frac{1}{3}b^{(1)} c^{(2)} \partial (T^{(1)} - T^{(2)}). \quad (24)$$

By construction,  $S_{\text{gf}}$  is invariant under the BRST transformations

$$\delta \Phi^A = \frac{\delta S[\Phi, \Phi^*]}{\delta \Phi_A^*} \Big|_{\Sigma} \lambda. \quad (25)$$

Integrating over the Nakanishi–Lautrup fields  $\pi_{(1)}$  and  $\pi_{(2)}$  imposes the gauge-fixing conditions  $h_{(2)} = \hat{h}_{(2)}$  and  $h_{(3)} = \hat{h}_{(3)}$ . The action is still BRST-invariant, but because of the elimination of  $\pi_{(1)}$ ,  $\pi_{(2)}$ ,  $h_{(2)}$  and  $h_{(3)}$ , the BRST transformation rules are modified by equations-of-motion terms. The BRST transformations are thus given by

$$\delta \varphi^{(1)} = (c^{(1)} - \frac{1}{3}b^{(1)} \partial c^{(2)} c^{(2)}) \lambda \partial \varphi^{(1)} + i c^{(2)} \lambda (\partial \varphi^{(1)} \partial \varphi^{(1)} - \frac{1}{3} \text{tr} \partial \varphi^{(1)} \partial \varphi^{(1)}), \\ \delta \varphi^{(2)} = (c^{(1)} + \frac{1}{3}b^{(1)} \partial c^{(2)} c^{(2)}) \lambda \partial \varphi^{(2)} \mp c^{(2)} \lambda (\partial \varphi^{(2)} \partial \varphi^{(2)} - \frac{1}{3} \text{tr} \partial \varphi^{(2)} \partial \varphi^{(2)}), \\ \delta c^{(1)} = -[c^{(1)} \partial c^{(1)} - \frac{1}{3} \partial c^{(2)} c^{(2)} (T^{(1)} - T^{(2)})] \lambda, \\ \delta c^{(2)} = -(2c^{(2)} \partial c^{(1)} + c^{(1)} \partial c^{(2)}) \lambda, \\ \delta b^{(1)} = -(T^{(1)} + T^{(2)} + T_{\text{gh}}) \lambda, \\ \delta b^{(2)} = -(W^{(1)} \pm iW^{(2)} + W_{\text{gh}}) \lambda, \quad (26)$$

and the corresponding *classical* BRST charge follows immediately: it is exactly that in (4), with  $T_{\text{gh}}$  and  $W_{\text{gh}}$  given in eq. (24). It is not difficult to verify that the Poisson bracket of two BRST charges vanishes modulo equations-of-motion terms. The fact that nilpotency of the BRST transformations holds only on-shell is a well-known fact for open-gauge algebras [15].

In order to promote the classical BRST charge to a charge which also closes at the quantum level, one may proceed as follows. One starts from the classical action given in (23) and calculates the possible anomalies that may arise in the quantization of this action. They occur in two types: universal anomalies and matter-dependent ones [16]. The cancellation of these possible anomalies leads to the addition of counter terms to the classical action. It turns out that these counterterms occur as renormalizations of the classical currents given in (10) and (24). The quantum BRST charge may then be obtained from the classical BRST charge, simply by replacing every classical current by the corresponding quantum expression. In other words, the cancellation of all anomalies in the quantum theory defined by the action (23) is equivalent to the construction of a nilpotent quantum BRST operator. This approach has been advocated in the context of ordinary  $W_3$  gravity in ref. [17].

In this letter, we will follow another, but equivalent, approach in which the relevant counter terms and corresponding renormalizations of the currents are derived from the requirement that the theory does not depend on the chosen gauge. We write  $S_0 = S^{(1)} + S^{(2)}$ , where the labels (1) and (2) refer to the matter and Liouville sector, respectively, and integrate out the Nakanishi–Lautrup fields in the classical gauge-fixed action (23). The condition that the theory does not depend on the chosen gauge leads to the requirement

$$\frac{\delta W[\hat{h}_{(2)}, \hat{h}_{(3)}]}{\delta \hat{h}_{(2)}} = \frac{\delta W[\hat{h}_{(2)}, \hat{h}_{(3)}]}{\delta \hat{h}_{(3)}} = 0, \quad (27)$$

where the effective action  $W[\hat{h}_{(2)}, \hat{h}_{(3)}]$  is defined by

$$\exp\{-W[\hat{h}_{(2)}, \hat{h}_{(3)}]\} = \left\langle \exp\left(-\frac{1}{\pi} \int [\hat{h}_{(2)}(T^{(1)} + T^{(2)} + T_{\text{gh}}) + \hat{h}_{(3)}(W^{(1)} \pm iW^{(2)} + W_{\text{gh}})]\right) \right\rangle. \quad (28)$$

To obtain the quantum expression for  $T^{(1)}$  and  $T^{(2)}$ , we first set  $\hat{h}_{(3)} = 0$  and find that

$$W[\hat{h}_{(2)}, 0] = \frac{c_1 + c_2 - 100}{24\pi} \Gamma[\hat{h}_{(2)}], \quad (29)$$

where  $\Gamma[\hat{h}_{(2)}]$  is Polyakov's action for induced gravity [18]. This leads to the condition that the central charges of the matter systems should add up to 100:

$$c_1 + c_2 = 100. \quad (30)$$

Clearly, the classical realizations of  $T^{(1)}$  and  $T^{(2)}$  given in eqs. (7) and (10) do not satisfy this requirement. We have to add counter terms to the classical action, which appear as renormalizations of the classical currents. In the case at hand, one finds that the quantum currents that satisfy the condition (30) are given by

$$T^{(i)} = -\frac{1}{2} \text{tr}(\partial\varphi^{(i)} \partial\varphi^{(i)}) + Q_i \text{tr}\left(\partial^2\varphi^{(i)} \sum_{r=1}^2 H_r\right),$$

$$W^{(i)} = -\frac{1}{3} i \text{tr}(\partial\varphi^{(i)} \partial\varphi^{(i)} \partial\varphi^{(i)}) + iQ_i(\partial^2\varphi_1^{(i)} \partial\varphi_1^{(i)}) - 2iQ_i(\partial^2\varphi_2^{(i)} \partial\varphi_2^{(i)}) - \frac{1}{2}Q_i \partial T^{(i)} + iQ_i^2 \partial^3\varphi_2^{(i)}, \quad (31)$$

where

$$Q_1 = i\left(\sqrt{t} + \frac{1}{\sqrt{t}}\right), \quad Q_2 = \left(\sqrt{t} - \frac{1}{\sqrt{t}}\right), \quad t \in \mathbb{R}. \quad (32)$$

These quantum currents form two commuting copies of the  $W_3$  algebra.

To find the appropriate renormalizations of the ghost currents, we next consider the term in  $W[\hat{h}_{(2)}, \hat{h}_{(3)}]$  with two external  $\hat{h}_{(3)}$  fields and one external  $\hat{h}_{(2)}$  field. This term vanishes if and only if  $W^{(1)} \pm iW^{(2)} + W_{\text{gh}}$  is primary with respect to  $T^{(1)} + T^{(2)} + T_{\text{gh}}$  at  $c_1 + c_2 = 100$ . Again, in order to achieve this, we must add local counter terms to the action. In this case they occur as renormalizations of the classical  $W_{\text{gh}}$  current of eq. (24). The corresponding quantum current is obtained by adding the following additional terms to the classical current:

$$\frac{17\beta_1^2 - 1}{90\beta_1^2} (2\partial^3 b^{(1)} c^{(2)} + 9\partial^2 b^{(1)} \partial c^{(2)} + 15\partial b^{(1)} \partial^2 c^{(2)} + 10b^{(1)} \partial^3 c^{(2)}). \quad (33)$$

Since all counter terms occur as renormalizations of the matter and ghost currents, the quantum effective action is obtained from the classical action by replacing the classical matter, Liouville and ghost currents, by the corresponding quantum expressions which are given in eqs. (31), (24) and (33). The quantum BRST charge is obtained by a similar replacement in the classical BRST charge. One can verify that, after this replacement,  $Q$  is indeed nilpotent. The quantum BRST charge obtained in this way is, up to a trivial rescaling  $W^{(i)} \rightarrow W^{(i)}/\sqrt{3}$ ,  $W_{\text{gh}} \rightarrow \pm iW_{\text{gh}}/\sqrt{3}$ ,  $h_{(3)} \rightarrow \sqrt{3}h_{(3)}$ ,  $c^{(2)} \rightarrow \mp i\sqrt{3}c^{(2)}$  and  $b^{(2)} \rightarrow \pm ib^{(2)}/\sqrt{3}$ , precisely equal to the one introduced in ref. [9]. The sign ambiguity in the second term of eq. (4) directly follows from the sign ambiguity in eq. (8). The above approach of deriving the counter terms resembles an approach that was performed for ordinary  $W_3$  gravity in refs. [16,19].



#### 4. Conclusions

In this paper we showed how to derive the BRST charge of ref. [9] from a lagrangian point of view. Our basic observation is that the system of Liouville and matter fields at the classical level is based upon a *closed* gauge algebra, which is a modification of the classical  $w_3$  algebra. Quantization of the theory then leads exactly to the expression of the quantum BRST charge given in ref. [9]. A novelty of this BRST charge is that nilpotency is achieved without the presence of a closed quantum algebra. It would be interesting to see whether the modified  $w_3$  algebra can nevertheless be extended to a quantum algebra in some way or another.

A priori the scalars considered in this paper are not necessarily designated to represent either matter or Liouville fields. Instead, one may consider the possibility that they can all be viewed as matter fields so that the  $W_3$  BRST charge of ref. [9] could be used to construct new critical  $W_3$  strings. An interesting possibility is the following. Take a multi-scalar realization of  $W_3$  [8]. As pointed out in ref. [8], only one of the scalars, say  $\rho$ , occurs explicitly. All other scalars occur via their energy-momentum tensor, say  $T_\mu$ . Furthermore, the contribution of the different scalars to the central charge  $c_1$  is given by  $c_\rho = \frac{3}{4}c_1 - \frac{1}{2}$  and  $c_\mu = \frac{1}{4}c_1 + \frac{1}{2}$ . Note that the critical value  $c_1 = 100$  corresponds to a non-critical value  $c_\mu = 25\frac{1}{2}$  of the Virasoro algebra. Therefore, it seems that one cannot embed a critical Virasoro string into a critical  $W_3$  string [8]. Interestingly enough, such an embedding is possible by combining the multi-scalar realization of ref. [8] with a one-scalar realization of  $W_3$  at  $c_2 = -2$  [20] to form a modified  $w_3$  algebra, as described in this paper. In that case the total central charge is given by  $c_1 - 2$ . Now the critical value  $c_1 - 2 = 100$  corresponds to the critical value  $c_\mu = 26$ . One thus obtains a critical  $W_3$  string, in which there exists a critical Virasoro string with 26 free scalars. The properties of this critical  $W_3$  string will be investigated elsewhere [21].

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